9 Optimal Detector for Waveform Channel

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Waveform Channel

$$R(t) = S(t) + \underbrace{N(t)}$$

↳ Noise model:

Additive white noise with PSD $\frac{N_0}{2}$

Goals: (1) Convert this model to $\vec{R} = \vec{S} + \vec{N}$

(2) How to implement the optimal receiver in practice

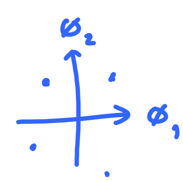
Recall: For $S(t)$, we have M choices:

$$s_1(t), s_2(t), \ldots, s_M(t)$$

↓ use GSOP

find orthonormal basis $\phi_1, \phi_2, \ldots, \phi_K$

$$\vec{s}^{(1)}, \vec{s}^{(2)}, \ldots, \vec{s}^{(M)}$$



Define $E_i = \langle s_i, s_i \rangle = \langle \vec{s}^{(i)}, \vec{s}^{(i)} \rangle$

$$p_i = P[S(t) = s_i(t)] = P[\vec{S} = \vec{s}^{(i)}] = P[W = i]$$

At the receiver, define $\vec{R}$ by
(receive $R(t)$)

↳ K dimensions

the j^{th} component: $R_j = \langle R(t), \phi_j(t) \rangle$

$$\vec{R} = \begin{pmatrix} \langle R(t), \phi_1(t) \rangle \\ \langle R(t), \phi_2(t) \rangle \\ \vdots \\ \langle R(t), \phi_K(t) \rangle \end{pmatrix}$$

$$R_j = \langle R(t), \phi_j(t) \rangle = \langle S(t) + N(t), \phi_j(t) \rangle$$

$$= \langle S(t), \phi_j(t) \rangle + \underbrace{\langle N(t), \phi_j(t) \rangle}_{\equiv}$$

$$= S_j + N_j$$

The j^{th} component of $\vec{S}$

So, we have $\vec{R} = \vec{S} + \vec{N}$.

$$R(t) = S(t) + N(t) = S(t) + \underbrace{\sum_{j=1}^{K} N_j \phi_j(t)}_{\tilde{N}(t)} + (N(t) - \tilde{N}(t))$$

$$= \sum_{j=1}^{K} S_j \phi_j(t) + \sum_{j=1}^{K} N_j \phi_j(t) + (N(t) - \tilde{N}(t))$$

$$= \sum_{j=1}^{K} R_j \phi_j(t) + \underbrace{(N(t) - \tilde{N}(t))}$$

Fact: This part does not contain any information about the transmitted signal and therefore has no effect in the detection process and can be ignored without sacrificing the optimality of the detector.
In other words, this part is irrelevant information for optimal detection.

Conclusion: For the design of the optimal detector

the waveform channel $R(t) = S(t) + N(t)$

is equivalent to

the K-dimensional vector channel $\vec{R} = \vec{S} + \vec{N}$.

Optimal Detector : MAP detector

$$\hat{w} = \arg\max_i \; p_i f_{\vec{N}}(\vec{r} - \vec{s}^{(i)})$$

For $\mathcal{N}$ noise,

$$\stackrel{\downarrow}{=} \arg\min_i \; \|\vec{r} - \vec{s}^{(i)}\|^2 - 2\sigma^2 \ln p_i \quad ; \; \sigma^2 = \frac{N_0}{2}$$

$$= \arg\max_i \; \langle \vec{r}, \vec{s}^{(i)} \rangle + \underbrace{\sigma^2 \ln p_i - \frac{1}{2} E_i}$$

$$\underbrace{\qquad\qquad\qquad}_{\text{bias term}}$$

η_i

Fact : $\langle x(t), s(t) \rangle = \langle \vec{x}, \vec{s} \rangle$ where $x_i = \langle x(t), \phi_i(t) \rangle$

How can we evaluate $\langle x(t), s(t) \rangle$ $\underbrace{\text{in practice}}$?

$\downarrow$

using filter

$$x(t) \longrightarrow \boxed{} \longrightarrow \langle x(t), s(t) \rangle = \int_{-\infty}^{\infty} x(t)\, s^*(t)\, dt$$

correlation

Filter $\longrightarrow$ convolution

$h(t)$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Let's try $h(t) = s^*(T-t)$ $\quad$ T is some constant.

$$h(t-\tau) = s^*\big(T-(t-\tau)\big) = s^*(T-t+\tau)$$

$$\{x * h\}(t) = \int_{-\infty}^{\infty} x(\tau)\, s^*(T-t+\tau)\, d\tau$$

$$\{x * h\}(T) = \int_{-\infty}^{\infty} x(\tau)\, s^*(\tau)\, d\tau = \langle x(t), s(t) \rangle$$

$$x(t) \longrightarrow \boxed{s^*(T-t)} \xrightarrow{t=T} \langle x(t), s(t) \rangle$$

This filter is called the "$\underbrace{\text{matched}}$ filter"

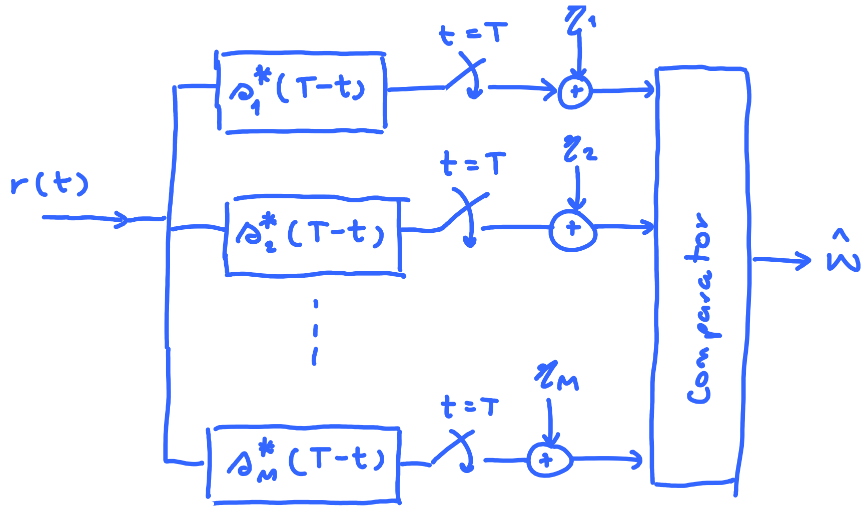
$\downarrow$

matched to $s(t)$

Conclusion : Implementation of optimal receiver (detector).

$$\longrightarrow \boxed{s^*(T-t)} \xrightarrow{t=T} \oplus \longrightarrow \boxed{}$$

η_i



r(t)
s*1(T-t)
s*2(T-t)
s*M(T-t)
t=T
t=T
t=T
η1
η2
ηM
Comparator
ŵ